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LETTER TO THE EDITOR

Improved mean-field theory of two-dimensional traffic flow models

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Abstract. An improved mean-field theory is proposed for traffic-flow models in two dimensions, both in the absence and presence of faulty traffic lights. The present theory includes the effects of blockage of cars due to cars moving in the same direction, while previous theories included only the effects of cars moving in the perpendicular direction. In the absence of faulty lights, the theory gives a much better estimate of the critical car density, above which the traffic is in a jamming phase. In the presence of faulty lights, the theory captures all the essential features in published simulation data. It gives the correct behaviour of the velocity in the dilute limit of car density. The dependence of the critical car density on the fraction of faulty traffic lights is studied.

Recently, there has been much interest in the study of traffic flow problems within the context of cellular-automaton (CA) models. These models have the advantages of being simple and easy to implement on computers, and can be readily generalized to study different effects on the road. The idea is similar to the application of CA models to other dynamical problems such as fluid flow. In one dimension (1D), Negal and Schrekenberg [1] introduced a stochastic discrete CA model to study the transition from laminar traffic flow to start–stop waves as the car density increases. Effects of bottleneck regions, slower and takeover sites, acceleration, exit and entrance sites, quenched disorderness, and separation-dependent car velocities, etc on 1D traffic flow have been studied [2–7]. Biham, Middleton and Levine [8] (henceforth referred to as BML) introduced a simple two-dimensional (2D) CA model with traffic lights and studied the average velocity of cars as a function of their density. In the original BML model, cars moving from west to east on a lattice will attempt a move in the first half of a time step, say, and cars heading from south to north will attempt a move in the second half of the time step. It was found that the average velocity in the long time limit vanishes when the concentration of cars is higher than a critical value. The basic BML model has been modified to study the effects of two-level crossing (overpasses), accidents on the road, slower sites, and faulty traffic lights [9–12].

The drop to vanishingly small velocity for car concentrations above a critical value signifies a transition from a moving phase for low car densities to a jamming phase for high car densities. Similar to dealing with phase transitions in equilibrium systems, mean-field theories have been useful in providing basic understanding of phenomena in dynamical

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systems. Several versions of mean-field theory have been proposed [13, 14]. For the basic BML model, Nagatani [13] has developed a mean-field treatment that takes into account the blockage of cars in one direction by cars moving in the perpendicular direction. He has also extended the method to anisotropic car distribution in which there are different densities for northbound and eastbound cars, and to cases of crossing of two perpendicular lanes. The treatments give qualitatively correct results when compared with simulation data. However, the theories do not properly account for the blockage of cars by cars moving in the *same* direction—an effect intrinsically built into the CA models and this is the most important factor in 1D models. This defect leads to an overestimate, among other parameters, of the critical car density.

In this letter, we introduce an improved mean-field theory for the BML model, and for BML model with faulty traffic lights. For the BML model, we found a value for the critical car density much closer to the numerical result. For the BML model with faulty traffic lights, the mean-field theory reproduces the features of the simulation result. The critical car density drops from $p_c = 0.343$ for the BML model monotonically to $p_c = 0.166$ to the case of 100% faulty traffic lights. Both of these values are in reasonable agreement with published simulation results [8].

In the basic BML model [8], cars are placed in an $N \times N$-block square lattice, with equal numbers of northbound and eastbound cars. Each block on the lattice can be in one of the three states: (i) occupied by a northbound car, (ii) occupied by an eastbound car, (iii) unoccupied. The updating rule is that northbound (eastbound) cars will only attempt a move in the first (second) half of a time step so as to regulate the flow to prevent two cars from moving into the same site simultaneously. Thus, the first and second halves of a time step represent the function of traffic lights. When a car attempts a move, it will move forward one step unless the site in front of it is blocked by another car. In that case, it will not move, even if the blocking car leaves on the same time step.

As the car distribution is isotropic, the velocities, which are defined as the ratio of number of steps that the cars moved to the total attempted moves in a time step, in the long time limit are identical in both directions and will be denoted by v. It takes on a maximum value of unity in the dilute limit of car density and vanishes in the jamming phase. Let p be the total car density, i.e. $p/2$ of northbound cars and $p/2$ of eastbound cars. Consider the velocity in the eastbound direction. This depends on two factors as an eastbound car may be blocked by a northbound car *or* by another eastbound car. As northbound cars spend on average a time $1/v$ on a site, they reduce the speed of eastbound cars from unity by $p/(2v)$. For eastbound cars, the extra amount of time that an eastbound car stays on a site is given by $1/v - 1$. This delay reduces the speed by an amount $\frac{1}{2}p(1/v - 1)$. Hence, a self-consistency equation for the speed v is

$$v = 1 - \frac{p}{2v} - \frac{p}{2}\left(\frac{1}{v} - 1\right). \tag{1}$$

The last term is the correction term taking the effects of blockage of cars moving in the same direction into account. It was not included in previously proposed mean-field theories [13]. Equation (1) is a quadratic equation for v and can be readily solved to give

$$v = \frac{1}{2} + \frac{p}{4} + \frac{1}{2}\sqrt{\frac{1}{4}(2+p)^2 - 4p} \tag{2}$$

for $p < p_c$, where the critical car density p_c is determined by the density at which equation (2) ceases to give a real solution and is given by

$$p_c = 6 - \sqrt{32} \approx 0.343. \tag{3}$$

Mean-field theory without the correction term [13] gives a value of $p_c = 1/2$; while simulation results given by BML [8] on 512×512 lattices give $p_c \sim 0.315$. Figure 1 shows the speed v as a function of p within the present theory. For comparsion, simulation results on 512×512 lattices given by BML [8] are sketched and the results of the mean-field theory without the correction term are also included. We note that for simulations on lattices of larger size, the critical car density is expected to shift to a smaller value. Our theory hence gives a much better estimate of the critical car density than the previous theory.

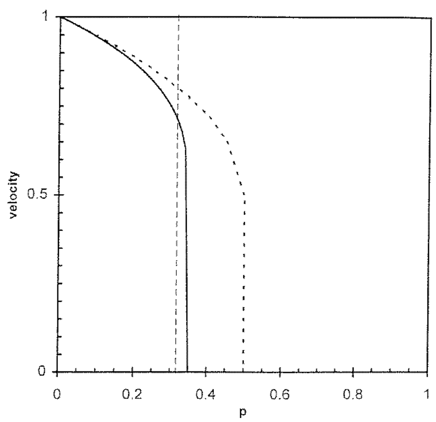

Figure 1. The velocity as a function of car density within the present theory (full curve). For comparsion, we sketched the results (long dashes) from [8] from simulations on 512×512 lattices and included results of previous mean-field theory (short dashes) from [13].

The present approach can be readily extended to include the effects of faulty traffic lights. Simulation data in the literature [12] show that faulty lights have two effects. At low car densities, intentionally turning off the traffic lights in a city may improve traffic flow in the sense that cars do not need to stop at lights. However, the critical car density is reduced by faulty traffic lights. Hence, a concentration originally in the moving phase within the BML model may become jammed with the introduction of faulty lights. We model the effects of faulty traffic lights as follows [12]. Let c be the fraction of faulty traffic lights. For an empty site with a faulty light, both northbound cars to the south and eastbound cars to the west of the site may attempt to enter, regardless of whether it is the first or second half of a time step. So, a northbound car to the south of a site with a faulty light will be able to move forward, even if it is the second half of the time step, provided that no eastbound car is trying to enter the site simultaneously. The same goes for an eastbound car to the west of the site with a faulty light. In the case of two cars simultaneously attempting to enter the site, one of them will be chosen randomly to enter the site, and the other car will not move at that attempt. With the introduction of faulty lights, it becomes possible for a car to move twice in one time step, resulting in a velocity larger than unity.

Consider the velocity in the eastbound direction. For those $1 - c$ fraction of sites with traffic lights, the effect is again given by equation (1). The presence of faulty lights leads to an additional term proportional to c in the self-consistency equation of the velocity. The maximum possible value of v is now two obtained in the limit of dilute car density and $c = 1$. For $0 < c < 1$ and given p, the effects of faulty lights can be best understood by considering the reduction in speed from its maximum possible value due to blockage by cars in either directions. The two halves of the time step are treated as independent of each other. The probability that a northbound car occupying a

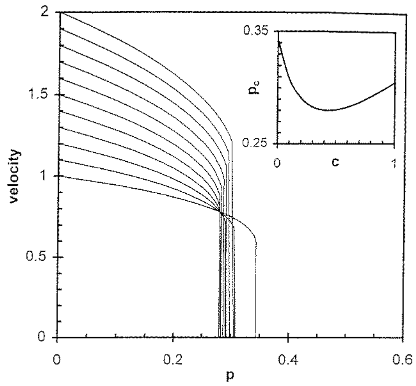

Figure 2. The velocity as a function of car density for different fractions of faulty traffic lights on the road within our mean field theory. The lines at $p = 0.1$, from bottom to top, correspond to the fractions $c = 0, 0.1, 0.2, \ldots, 1.0$. The critical car density drops as the fraction of faulty lights increases. The inset shows the dependence of the critical car density on the fraction of faulty lights.

site with faulty light and blocking the eastbound car from moving is $2(p/2)(1/v) = p/v$. Similarly, the probability that another eastbound car occupying the site with a faulty light and preventing an eastbound car from moving forward is $2(p/2)(1/v-1/2) = p(1/v-1/2)$. Putting these considerations together, if an eastbound car wins in an attempt to move into a site with faulty lights, its speed is reduced from its possible maximum value of two to $2(1-p/v-p(1/v-1/2)) = 1-2p/v+p/2$. However, an eastbound car has a probability $(1-p/4)$ of getting into an empty site with faulty light due to the competition of northbound car trying to get into the same site. Hence the inclusion of faulty traffic lights leads to a modification of equation (1) of

$$v = (1-c)\left(1-\frac{p}{v}+\frac{p}{2}\right)+2c\left(1-\frac{p}{4}\right)\left(1-\frac{2p}{v}+\frac{p}{2}\right) \tag{4}$$

which gives a quadratic equation for v

$$v^2-\left(1+c+\frac{cp}{2}\right)\left(1+\frac{p}{2}\right)v+(1+3c-cp)p = 0. \tag{5}$$

Equation (5) can be solved for v for $p < p_c$, which p_c determined by setting the discriminant to zero. Thus p_c satisfies the equation

$$c^2p_c^4-4cp_c^3+4(1+13c-2c^2)p_c^2-16(3+11c)p_c+16(1+c)^2 = 0. \tag{6}$$

For $c = 0$, equation (4) reduces to equation (1), and equation (6) gives the previous result $6-\sqrt{32}$. For $c = 1$, i.e. when all the traffic lights are turned off, equation (6) is a quartic equation for p_c and gives $p_c \approx 0.307$. For this case, BML [8] reported a value of $p_c \sim 0.1$ obtained from simulations on 512×512 lattices. Figure 2 shows the velocity as a function p for different fractions of faulty traffic lights. The features are in reasonable agreement with simulation results obtained by Hui and coworkers [12]. At low car densities, the average speed increases with the introduction of faulty lights as cars may move continuously without stopping at traffic lights. In the dilute ($p \to 0$) limit, the present theory gives $v = 1+c$, in agreement with numerical results. The inset shows the dependence of p_c on c. The critical car density drops from 0.343 at $c = 0$ to its minimum value of about 0.280 at $c = 0.432$ and then increases to 0.307 for $c = 1$. Such a dependence can be understood as the general effects of inhomogenities in an otherwise homogeneous medium, although the role of inhomogenenity is played by the faulty lights in the $c \to 0$ limit, and by the traffic lights in the $c \to 1$ limit. Near $c = 0$, a moving pattern at car density just below the critical car density is disrupted by the introduction of faulty traffic lights and it is thus harder to form a moving pattern. Near $c = 1$, a moving pattern at car density just below the critical car density is slowed down by the introduction of normal traffic lights, which act as seeds for forming queues as cars have to stay there for one more step. Thus the addition of inhomogeneities in either limit leads to a reduction in the average speed. Simulation data in the literature [12] on 128×128 lattices show some evidence of this feature. However, it is difficult to precisely determine p_c from simulations on small lattices, both due to finite size effect and to the necessity of averaging a large number of initial configurations to obtain good statistics.

In summary, we presented an improved mean-field theory for 2D traffic-flow models without and with faulty traffic lights. The theory gives better estimates for the critical car densities and captures the essential features in the relation between the average velocity and car density. The present theory can be readily generalized to problems with a combination of the following possibilities on road condition including anisotropic distributions of cars in the two directions, overpasses and/or slower sites on the roads, faulty traffic lights, etc. As numerical simulation data are not available in the literature for many of these cases, we shall defer the publication of results in these situations to future work together with numerical results.

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References

[1] Nagel K and Schreckenberg M 1992 *J. Physique* I **2** 2221
[2] Yukawa S, Kikuchi M and S Tadaki 1994 *J. Phys. Soc. Japan* **63** 3609
[3] Chung K H and Hui P M 1994 *J. Phys. Soc. Japan* **63** 4338
[4] Benjamin S C, Johnson N F and Hui P M 1995 *Preprint*
[5] Csahók Z and Vicsek T 1994 *J. Phys. A: Math. Gen.* **27** L591
[6] Nagatani T 1993 *J. Phys. A: Math. Gen.* **26** 6625
[7] Janowshy S A and Lebowitz J L 1992 *Phys. Rev.* A **45** 618
[8] Biham O, Middleton A A and Levine D 1992 *Phys. Rev.* A **46** R6124
[9] Nagatani T 1993 *Phys. Rev.* E **48** 3290
[10] Gu G Q, Chung K H and Hui P M 1995 *Physica* **217A** 339
[11] Cuesta J A, Martinez F C, Movea J M and Sánchez 1993 *Phys. Rev.* E **48** R4175
[12] Chung K H, Hui P M and Gu G Q 1995 *Phys. Rev.* E **51** 772
[13] Nagatani T 1993 *J. Phys. Soc. Japan* **62** 2656
[14] Ishibashi Y and Fukui M 1994 *J. Phys. Soc. Japan* **63** 2882
[15] Nagatani T and Seno T 1994 *Physica* **207A** 574